

ESSENTIAL CENTRAL RANGE AND SELFADJOINT COMMUTATORS IN PROPERLY INFINITE VON NEUMANN ALGEBRAS⁽¹⁾

BY

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ABSTRACT. The essential central range of an element A of a von Neumann algebra with respect to a central ideal is characterized as those elements arbitrarily close to the compression of A to a subspace large with respect to the ideal. The selfadjoint commutators in a properly infinite algebra are shown to be the elements whose essential central ranges with respect to the strong radical contain 0.

1. Introduction. Let $\mathcal{Q}$ be a von Neumann algebra with center $\mathcal{Z}$. Let $(\mathcal{S})$ denote the set of nonzero projections in a subset $\mathcal{S}$ of $\mathcal{Q}$. A (closed two-sided) ideal $\mathcal{I}$ of $\mathcal{Q}$ is said to be a *central ideal* of $\mathcal{Q}$ if given any bounded set $\{A_i\}$ in $\mathcal{I}$ and a corresponding set $\{P_i\}$ of orthogonal projections in $(\mathcal{Z})$ then the operator $\sum A_i P_i$ is in $\mathcal{I}$. For an element ζ of the spectrum Z of $\mathcal{Z}$, let $[\zeta]$ denote the smallest ideal of $\mathcal{Q}$ containing ζ . Then the ideal $\mathcal{I}$ is a central ideal if and only if the map

$$\zeta \rightarrow \|A(\mathcal{I} + [\zeta])\| = \text{glb } \{\|A + B\| \mid B \in \mathcal{I} + [\zeta]\}$$

is continuous on Z for every A in $\mathcal{Q}$. If P is a projection in $\mathcal{Z}$ and if E is a properly infinite projection in $\mathcal{Q}P$ (by convention 0 is properly infinite), then the set of projections F in $\mathcal{Q}P$ such that $QF \succ EQ$ for some central projection Q implies $EQ = 0$ is the set of all projections of a central ideal denoted by $\mathcal{I}_P(E)$. All central ideals are of this form. If $\mathcal{Q}P$ is the weak closure of the central ideal, then the form is *canonical* in the sense that $\mathcal{I}_P(E) = \mathcal{I}_Q(F)$ if and only if $P = Q$ and $E \sim F$ [12].

Now let $\mathcal{I}$ be a central ideal of $\mathcal{Q}$. If A is an element in $\mathcal{Q}$, there is a largest central projection $c_{\mathcal{I}}(A)$ in $\mathcal{Q}$ such that $c_{\mathcal{I}}(A)A$ is in $\mathcal{I}$. The projection $1 - c_{\mathcal{I}}(1)$ is denoted by $P_{\mathcal{I}}$. A projection F in $\mathcal{Q}$ is said to have *dimension greater* than the central ideal $\mathcal{I} = \mathcal{I}_P(E)$ (in symbols, $\dim F > \dim \mathcal{I}$) if F has central support $P_{\mathcal{I}}$ and if $F \succ EP_{\mathcal{I}}$ or equivalently, if F has central support $P_{\mathcal{I}}$ and $c_{\mathcal{I}}(F)$

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$= 1 - P_g$. The *essential central spectrum* $\mathfrak{Z} - \text{Sp}_g A$ of an element A in $\mathcal{Q}$ modulo the central ideal $\mathfrak{g}$ is the set of all C in $\mathfrak{Z}$ such that $\hat{C}(\zeta)$ is in the spectrum of the image $A(\mathfrak{g} + [\zeta])$ of A in $\mathcal{Q}/(\mathfrak{g} + [\zeta])$ under the canonical homomorphism for every ζ in Z . Here $\hat{C}$ is the Gelfand transform of C . If $\mathfrak{g}$ is the ideal (0) , then the essential central spectrum of A modulo $\mathfrak{g}$ is called the *central spectrum* of A and is written $\mathfrak{Z} - \text{Sp } A$. The set $\mathfrak{Z} - \text{Sp}_g A$ is nonvoid and strongly closed. Let $\mathcal{Q}^\sim$ be the Banach space of all bounded $\mathfrak{Z}$ -module homomorphisms of $\mathcal{Q}$ into $\mathfrak{Z}$ and let $\mathcal{Q}^{\sim+}$ be the subset of all those ϕ in $\mathcal{Q}^\sim$ that map $\mathcal{Q}^+$ into $\mathfrak{Z}^+$. An element ϕ in $\mathcal{Q}^{\sim+}$ is called a *state* in $\mathcal{Q}^\sim$ if $\phi(1) = 1$. For example, if E is a *maximal abelian projection* (i.e., E has central support 1) in the commutant $\mathfrak{Z}'$ of $\mathfrak{Z}$ and if $\tau_E(A)$ denotes the unique element in $\mathfrak{Z}$ with $\tau_E(A)E = EAE$ for A in $\mathcal{Q}$, then the map τ_E is a state of $\mathcal{Q}^\sim$. Let $E_a(\mathfrak{g})$ be the subset of $\mathcal{Q}^{\sim+}$ given by

$$E_a(\mathfrak{g}) = \{\phi \in \mathcal{Q}^{\sim+} | \phi(\mathfrak{g}) = 0, \phi(P_g) = P_g\}.$$

The *essential central range* $\mathfrak{K}_g(A)$ of an element A in $\mathcal{Q}$ modulo the central ideal $\mathfrak{g}$ is defined to be the set

$$\mathfrak{K}_g(A) = \{\phi(A) | \phi \in E_a(\mathfrak{g})\}.$$

The set $\mathfrak{K}_g(A)$ is also equal to the set

$$\cap \{\text{unif clos } \{\tau_E(A + B) | E \text{ abelian of central support } P_g\} | B \in \mathfrak{g}\}$$

[12, 4.8], and is a weakly closed $\mathfrak{Z}$ -convex subset of $\mathfrak{Z}$ containing $\mathfrak{Z} - \text{Sp}_g A$. Here a set $\mathfrak{S}$ is $\mathfrak{Z}$ -convex if $CA_1 + (1 - C)A_2$ is in $\mathfrak{S}$ whenever C is in $\mathfrak{Z}$ with $0 \leq C \leq 1$ and the A_i are in $\mathfrak{S}$ [12].

If $\mathcal{Q}$ is a finite von Neumann algebra, then every central ideal is of the form $\mathcal{Q}P$ for some central projection P . If $\mathcal{Q}$ is a properly infinite von Neumann algebra, then the ideal $\mathfrak{g}_0$ generated by the finite projections is a central ideal and $\mathfrak{g}_0$ has canonical form

$$\mathfrak{g}_0 = \mathfrak{g}_1(E)$$

where E is a projection of central support 1 for which there is an orthogonal set $\{P_i\}$ in $(\mathfrak{Z})$ of sum 1 such that EP_i is a σ -finite projection. If $\mathcal{Q}$ is the set of all bounded operators on a Hilbert space, then $\mathfrak{g}_0$ is the ideal of compact operators. In general, we have that $\mathfrak{K}_{\mathfrak{g}_0}(A)$ is equal to the intersection of $\mathfrak{Z}$ with the weak closure of the convex hull of the set UAU^* where U runs through the set $U(\mathcal{Q})$ of unitary operators in $\mathcal{Q}$. The *strong radical* $\mathfrak{g}$ of $\mathcal{Q}$ (i.e., the intersection of all maximal ideals of $\mathcal{Q}$) is also a central ideal of $\mathcal{Q}$ and $\mathfrak{g}$ may be written as $\mathfrak{g} = \mathfrak{g}_1(1)$. In particular, the dimension of the projection E is greater than that of $\mathfrak{g}$ if and only if $E \sim 1$. In this case the set $\mathfrak{K}_g(A)$ is the

intersection of $\mathfrak{Z}$ with the uniform closure of the convex hull of $\{UAU^* | U \in U(\mathcal{Q})\}$. In the sequel, we write $\mathcal{K}_{\mathfrak{g}}(A)$ simply as $\mathcal{K}(\mathcal{Q})$ [10], [12].

In this note we show that C is in the essential central range of A modulo the central ideal $\mathfrak{g}$ if and only if there is, for every $\varepsilon > 0$, a projection E in $\mathcal{Q}$ with $\dim E > \dim \mathfrak{g}$ such that $\|E(C - A)E\| < \varepsilon$. For a properly infinite algebra $\mathcal{Q}$, the operator C is in $\mathcal{K}_{\mathfrak{g}}(A)$ if and only if there is a projection E in $\mathcal{Q}$ with $\dim E > \dim \mathfrak{g}$ such that CE is in the essential central spectrum of EAE in the algebra $\mathcal{Q}_E$ modulo the central ideal $\mathfrak{g} \cap \mathcal{Q}_E$. Here $\mathcal{Q}_E$ is the von Neumann algebra $E\mathcal{Q}E$ on the Hilbert space determined by E . Also the set $\mathcal{K}_{\mathfrak{g}_0}(A)$ is shown to be the intersection of the center with the weak closure of $\{UAU^* | U \in U(\mathcal{Q})\}$. Using this characterization of the essential central range, we show by displaying a matrix form similar to that given by J. Anderson [1] that an element A in a properly infinite von Neumann algebra $\mathcal{Q}$ is a *selfadjoint commutator* in $\mathcal{Q}$ (i.e., there are B and C in $\mathcal{Q}$ with $A = BC - CB$ and $B = B^*$) if and only if 0 is in $\mathcal{K}(A)$. We also characterize operators similar to selfadjoint commutators. Finally, we discuss the problem of characterizing commutators in properly infinite von Neumann algebras.

2. Characterization of the essential central range. In this section we show that a central element C is in the essential range of A modulo a central ideal $\mathfrak{g}$ if and only if the compression of $C - A$ to a subspace that is large with respect to $\mathfrak{g}$ is small.

LEMMA 2.1. *Let C be in the essential central range of an element A in a continuous von Neumann algebra $\mathcal{Q}$ modulo the central ideal $\mathfrak{g}$ of $\mathcal{Q}$; then there is, for every $\varepsilon > 0$, an extreme point ϕ of $E_a(\mathfrak{g})$ such that $\|\phi(A) - C\| < \varepsilon$ and such that the kernel of the canonical representation of the positive functional $\phi_{\mathfrak{g}}$ given by $\phi_{\mathfrak{g}}(B) = \phi(B)^{\wedge}(\zeta)$ is $\mathfrak{g} + [\zeta]$ for every ζ in the spectrum Z of the center $\mathfrak{Z}$ of $\mathcal{Q}$.*

PROOF. By performing a preliminary reduction, we may assume that $P_{\mathfrak{g}} = 1$ so that the canonical representation $\mathfrak{g}_P(E)$ of $\mathfrak{g}$ has the property that the central support of E is P [11, 2.7]. The set $\mathfrak{S}$ of all positive $\mathfrak{Z}$ -module homomorphisms ψ of $\mathcal{Q}$ into $\mathfrak{Z}$ such that $\psi(\mathfrak{g}) = (0)$ and $\psi(E) = P$ is a nonvoid, $\mathfrak{Z}$ -convex subset of $\mathcal{Q}^{\sim+}$ that is compact in the topology of pointwise convergence on $\mathcal{Q}$ where $\mathfrak{Z}$ is taken with the weak operator topology [12, 4.3]. If ψ_1 is an extreme point of $\mathfrak{S}$ and if ψ_2 is a pure (i.e., extreme) state of $\mathcal{Q}^{\sim+}$ whose canonical representation is faithful [9, 4.6], then the $\mathfrak{Z}$ -module homomorphism $\phi = P\psi_1 + (1 - P)\psi_2$ is a pure state of $\mathcal{Q}^{\sim+}$. We show that the kernel of the canonical representation $\pi_{\mathfrak{g}}$ induced by the state $\phi_{\mathfrak{g}}$ is $\mathfrak{g} + [\zeta]$. On the one hand, if $P^{\wedge}(\zeta) = 0$, then $\pi_{\mathfrak{g}}(B) = 0$ if and only if

$$(\psi_2)_{\mathfrak{g}}(D^*BD) = \phi_{\mathfrak{g}}(D^*BD) = 0$$

for all D in $\mathcal{Q}$, or equivalently, if and only if B is in the kernel $[\zeta] = \mathfrak{g} + [\zeta]$ of the canonical representation of $(\psi_2)_\zeta$ [9, Theorem 4.7]. On the other hand, if $\hat{P}(\zeta) = 1$, then it is clear that the ideal $\mathfrak{g} + [\zeta]$ is contained in the kernel of π_ζ . Now supposing B is not in $\mathfrak{g} + [\zeta]$, we may find a spectral projection F of B^*B corresponding to an interval of the form $[\alpha, \|B\|^2]$ with $\alpha > 0$ such that F is not in $\mathfrak{g} + [\zeta]$. We have that $c_g(F)^\wedge(\zeta) = 0$; otherwise, the projection

$$F = c_g(F)F + (1 - c_g(F))F$$

is in $\mathfrak{g} + [\zeta]$. Because

$$(1 - c_g(F))F \succ (1 - c_g(F))E,$$

there is a partial isometry U in $\mathcal{Q}$ with $U^*U = (1 - c_g(F))E$ and $UU^* \leq (1 - c_g(F))F$. The fact that

$$\phi_\zeta(U^*U) = \phi_\zeta(E) = 1$$

implies that $\pi_\zeta(U) \neq 0$ and consequently that

$$\pi_\zeta(B^*B) \geq \alpha\pi_\zeta(F) \geq \alpha\pi_\zeta(UU^*) > 0.$$

This proves that the kernel of π_ζ is $\mathfrak{g} + [\zeta]$.

Now let C be in $\mathcal{K}_g(A)$. There is a ψ in $E_a(\mathfrak{g})$ with $\psi(A) = C$. For every ζ in Z , there is a state θ_ζ of $\pi_\zeta(\mathcal{Q})$ such that $\psi_\zeta = \theta_\zeta\pi_\zeta$. Since π_ζ is an irreducible representation of the continuous algebra $\mathcal{Q}$ [9, Corollary, Theorem 4.3], there is a unitary operator U in $\mathcal{Q}$ such that

$$(1) \quad |\theta_\zeta(\pi_\zeta(A)) - (\pi_\zeta(A)\pi_\zeta(U)x_\zeta, \pi_\zeta(U)x_\zeta)| < \varepsilon$$

[8, Theorem 2]. Here x_ζ denotes a unit vector in the Hilbert space of π_ζ such that $(\pi_\zeta(B)x_\zeta, x_\zeta) = \phi_\zeta(B)$ for every B in $\mathcal{Q}$. Due to the continuity of the left hand side of (1) as a function of ζ , we may use the fact that Z is extremally disconnected to find a unitary U in $\mathcal{Q}$ such that $\|\psi(A) - \phi(U^*AU)\| < \varepsilon$. The state $B \rightarrow \phi(U^*BU)^\wedge(\zeta)$ of $\mathcal{Q}$ is a pure state whose canonical representation has kernel $\mathfrak{g} + [\zeta]$. Q.E.D.

We now give a characterization of the essential central range. This was previously known only for selfadjoint elements [12, 4.9].

THEOREM 2.2. *In order that an element C in the center $\mathfrak{Z}$ of a von Neumann algebra $\mathcal{Q}$ be in the essential central range of an element A in $\mathcal{Q}$ with respect to the central ideal $\mathfrak{g}$, a necessary and sufficient condition is that $CP_g = C$ and that, for every $\varepsilon > 0$, there is a projection F in $\mathcal{Q}$ with $\dim F > \dim \mathfrak{g}$ such that $\|F(C - A)F\| < \varepsilon$.*

PROOF. Suppose first that C is in $\mathcal{K}_g(A)$. It is clear from the definition of $\mathcal{K}_g(A)$ that $CP_g = C$. There is no loss of generality in assuming that $C = 0$ and $P_g = 1$, and that $\mathcal{Q}$ is either a continuous, or a discrete finite or properly infinite algebra.

First we assume $\mathcal{Q}$ is continuous. Given $\varepsilon > 0$, there is an extreme point ϕ of $E_a(\mathcal{G})$ such that $\|\phi(A)\| < \varepsilon/3$, and such that the kernel of the canonical representation π_ζ induced by ϕ_ζ on the Hilbert space H_ζ is $\mathcal{G} + [\zeta]$ for every ζ in the spectrum Z of $\mathcal{Z}$ (Lemma 2.1). Let $D = A - \phi(A)$ and let $\{e_i | i \in I\}$ be a complete orthonormal base of H_ζ so that

$$\phi_\zeta(B) = (\pi_\zeta(B)e_1, e_1)$$

for every B in $\mathcal{Q}$. Here we are assuming $1 \in I$. For every S in the family $\mathcal{F}(I)$ of all finite subsets of $I - \{1\}$ directed by the inclusion relation, we may find a projection $E(S)$ in $\mathcal{Q}$ such that

$$\pi_\zeta(E(S))e_1 = e_1 \quad \text{and} \quad \pi_\zeta(E(S))e_i = 0 \quad (i \in S)$$

due to the fact π_ζ is irreducible (Lemma 2.1). Because the bounded net $\{\pi_\zeta(E(S)DE(S)) | S \in \mathcal{F}(I)\}$ converges to 0 in the strong operator topology on H_ζ , there is a set S in $\mathcal{F}(I)$ with

$$(2) \quad \|\pi_\zeta(E(S)DE(S))e_1\| < \varepsilon/3.$$

The projection $c_g(E)$, where $E = E_\zeta$ is the spectral projection of $E(S)D^*E(S)DE(S)$ in the von Neumann algebra $\mathcal{Q}_{E(S)}$ corresponding to the interval $[0, \varepsilon^2/3]$, is in the ideal ζ ; otherwise, the projection E is in the kernel $\mathcal{G} + [\zeta]$ of π_ζ and a relation

$$\begin{aligned} & \pi_\zeta(E(S)D^*E(S)DE(S)) \\ &= \pi_\zeta((E(S) - E)E(S)D^*E(S)DE(S)(E(S) - E)) \\ &\geq 3^{-1}\varepsilon^2\pi_\zeta(E(S) - E) \geq 3^{-1}\varepsilon^2\pi_\zeta(E(S)) \end{aligned}$$

incompatible with the relation (2) arises. Since the map

$$\xi \rightarrow \|\pi_\xi(EDE)\| = \|EDE(\mathcal{G} + [\xi])\|$$

is continuous on Z , we can use the fact that Z is extremally disconnected to find a projection E in $\mathcal{Q}$ such that $c_g(E) = 0$ and such that

$$\|EDE(\mathcal{G})\| = \text{lub } \{\|EDE(\mathcal{G} + [\zeta])\| | \zeta \in Z\} < \varepsilon/3$$

(cf. [12, 3.1]). If B is an element in $\mathcal{G}$ with $EBE = B$ and

$$\|EDE - B\| < \varepsilon/3,$$

and if F is a projection in $\mathfrak{G}$ majorized by E such that

$$\|B(E - F)\| < \varepsilon/3,$$

then we have that $c_{\mathfrak{G}}(E - F) = 0$, or equivalently, that $\dim(E - F) > \dim \mathfrak{G}$, and that

$$\|(E - F)A(E - F)\| \leq \|EDE - B\| + \|B(E - F)\| + \|\phi(A)\| < \varepsilon.$$

Now assume that $\mathcal{Q}$ is a type I algebra. There is no loss of generality in the assumption that the commutant $\mathcal{Q}'$ of $\mathcal{Q}$ is equal to $\mathfrak{Z}$. If $\mathcal{Q}$ is a finite algebra, the assumption that $P_{\mathfrak{G}} = 1$ implies $\mathfrak{G} = (0)$ [12, 2.3]. This means that there is a maximal abelian projection E of $\mathcal{Q}$ such that

$$\|EAE\| = \|\tau_E(A)E\| = \|\tau_E(A)\| < \varepsilon$$

[12, 4.8]. If $\mathcal{Q}$ is properly infinite type I algebra, let $\{E_i\}$ be a maximal set of nonzero orthogonal abelian projections in $\mathcal{Q}$ such that

$$\|E_iAE_j\| \leq 2^{-1}\delta_{ij}\varepsilon$$

for all i, j . Here δ_{ij} is the Kronecker delta. Setting $E = \sum E_i$, we get that

$$\|EAE\| = \text{lub } \|E_iAE_i\| \leq \varepsilon/2.$$

We complete the first part of the proof by showing that $c_{\mathfrak{G}}(E) = 0$. On the contrary, if $P = c_{\mathfrak{G}}(E)$ is nonzero, then the least upper bound F of the range projections of EP , AEP , and A^*EP is in $\mathfrak{G}$ [7, III, 1, Proposition 2]. If ϕ is $E_a(\mathfrak{G})$ with $\phi(A) = 0$, then the map ψ given by

$$\psi(B) = (P - F)\phi(B)$$

is a positive $\mathfrak{Z}(P - F)$ -module homomorphism of the von Neumann algebra $\mathcal{Q}_{(P-F)}$ into its center $\mathfrak{Z}(P - F)$ which vanishes on the central ideal $\mathfrak{G}_1 = \mathfrak{G} \cap \mathcal{Q}_{(P-F)}$ of $\mathcal{Q}_{(P-F)}$ and takes $P_{\mathfrak{G}_1} = P - F$ into $P - F$. There is a nonzero abelian projection G in $\mathcal{Q}_{(P-F)}$ such that

$$\|GAG\| \leq \varepsilon/2$$

due to the fact that

$$\psi((P - F)A(P - F)) = 0$$

implies that 0 is in $\mathfrak{K}_{\mathfrak{G}_1}((P - F)A(P - F))$. Since G is also a nonzero abelian projection in $\mathcal{Q}$ and since

$$GE_i = GAE_i = E_iAG = 0$$

for all i , the set $\{E_i\}$ is not maximal. This is a contradiction. Hence, we have that $c_g(E) = 0$.

The converse is already known [12, 4.9]. Q.E.D.

REMARK. If C is in the essential central range of an operator A with respect to the strong radical in a properly infinite algebra $\mathcal{Q}$, then we have shown elsewhere [11, Proof of Theorem 4] that there is an invertible B in $\mathcal{Q}$ and a projection E equivalent to 1 such that $EB^{-1}ABE = CE$.

The next result answers a question of L. Zsidó [15].

COROLLARY 2.3. *Let A be an element in a properly infinite von Neumann algebra $\mathcal{Q}$ with center $\mathfrak{Z}$; then the intersection of $\mathfrak{Z}$ with the sets*

$$\mathfrak{S}_1 = \text{weak closure} [\text{convex} \{UAU^* | U \in U(\mathcal{Q})\}]$$

and

$$\mathfrak{S}_2 = \text{weak closure} \{UAU^* | U \in U(\mathcal{Q})\}$$

coincide. Here $U(\mathcal{Q})$ is the set of unitary operators of $\mathcal{Q}$.

PROOF. The ideal $\mathfrak{G}_0$ of $\mathcal{Q}$ generated by the finite projections of $\mathcal{Q}$ is a central ideal and $\mathfrak{S}_1 \cap \mathfrak{Z}$ is equal to $\mathfrak{K}_{\mathfrak{G}_0}(A)$ [12, 4.16]. If C is in $\mathfrak{K}_{\mathfrak{G}_0}(A)$, then there is, for every $n = 1, 2, \dots$, a projection E_n with $\dim E_n > \dim \mathfrak{G}_0$ such that

$$\|E_n(C - A)E_n\| < n^{-1}$$

(Theorem 2.2). Each projection E_n is properly infinite with central support 1. We may assume that, for each n , there is a set $\{P_i | i \in I\}$ orthogonal central projections of sum 1 such that each P_i is the sum of an infinite set $\{E_{ij} | j \in J_i\}$ equivalent orthogonal projections such that $E_{ij} = E_n P_i$ for some j in J_i [7, III, 8, Theorem 1, Corollary 2 and III, 1, Theorem 1, Corollary 1]. Let $\mathfrak{X}$ be a finite set of unit vectors in the Hilbert space of $\mathcal{Q}$. There is a finite subset S of I , and for each $i \in S$, a finite subset S_i of J_i such that

$$\|(1 - \sum \{P_i E_{ij} | j \in S_i, i \in S\})x\| \leq (2(\|C - A\| + 1)n)^{-1}$$

for all x in $\mathfrak{X}$. There is, for every i in S , a unitary operator U_i in $\mathcal{Q}P_i$ that carries $F_i = E_n P_i$ onto $G_i = \sum \{P_i E_{ij} | j \in S_i\}$ and $P_i - F_i$ onto $P_i - G_i$ because $F_i \sim G_i$ and $P_i - F_i \sim P_i - G_i$. Thus there is a unitary operator $U = U(\mathfrak{X}, n)$ in $\mathcal{Q}$ such that $UP_i = U_i$ for $i \in S$. Setting $G = \sum \{G_i | i \in S\}$ and $P = \sum \{P_i | i \in S\}$, we get that

$$\begin{aligned}
& |((C - A)Ux, Ux)| \\
& \leq 2\|(C - A)\| \|(1 - G)x\| + |((C - A)E_n P Ux, E_n P Ux)| \\
& \leq n^{-1} + n^{-1} = 2n^{-1}
\end{aligned}$$

for every $x \in \mathfrak{X}$. This proves that C is in the set $\mathfrak{S}_2 \cap \mathfrak{Z}$. Since the relation $\mathfrak{S}_2 \cap \mathfrak{Z} \subset \mathfrak{S}_1 \cap \mathfrak{Z}$ is clear, we see that $\mathfrak{S}_1 \cap \mathfrak{Z} = \mathfrak{S}_2 \cap \mathfrak{Z}$. Q.E.D.

We now prove a stronger version of Theorem 2.2 that will be needed to characterize selfadjoint commutators. For this we need to study the spectral resolution of normal operators in a von Neumann algebra.

DEFINITION 2.4. Let Ω be the spectrum of the abelian von Neumann algebra $\mathfrak{B}$ generated by the center $\mathfrak{Z}$ of a von Neumann algebra $\mathfrak{Q}$ and a selfadjoint element A in $\mathfrak{Q}$; for every element C in the set $\mathfrak{Z}_h$ of selfadjoint elements of $\mathfrak{Z}$, let $E_A(C)$ denote the projection in $\mathfrak{B}$ whose Gelfand transform is the characteristic function of the open and closed subset closure $\{\omega \in \Omega | A^\wedge(\omega) < C^\wedge(\omega)\}$ of Ω .

We notice that $\{E_A(\lambda) | -\infty < \lambda < \infty\}$ is the spectral resolution of A .

PROPOSITION 2.5. Let A be a selfadjoint operator in the von Neumann algebra $\mathfrak{Q}$ with center $\mathfrak{Z}$ and let $\mathfrak{I}$ be a central ideal of $\mathfrak{Q}$. The map

$$C \rightarrow E_A(C) = E(C)$$

of $\mathfrak{Z}_h$ into the lattice of projections of the von Neumann algebra $\mathfrak{B}$ generated by $\mathfrak{Z}$ and A satisfies the following properties:

- (i) $PE(C) = PE(CP)$ for every C in $\mathfrak{Z}_h$ and every projection P in $\mathfrak{Z}$;
- (ii) $E(\text{lub } \mathfrak{X}) = \text{lub } \{E(C) | C \in \mathfrak{X}\}$ for every subset $\mathfrak{X}$ of $\mathfrak{Z}_h$ that is bounded above;
- (iii) $(A - C)(\text{glb } \{E(C') | C' > C\} - E(C)) = 0$ for every C in $\mathfrak{Z}_h$; and
- (iv) $c_{\mathfrak{I}}(E(C + D) - E(C - D)) = 0$ for every $D > 0$ in $\mathfrak{Z}_h$ if and only if $C \in \mathfrak{Z} - \text{Sp}_{\mathfrak{I}}A$.

Here $C > D$ means that the set $\{\omega \in \Omega | C^\wedge(\omega) > D^\wedge(\omega)\}$ is dense in Ω .

PROOF. (i) Clear.

(ii) This is clear if $\mathfrak{X}$ is a monotonely increasing net in $\mathfrak{Z}$. In general, for any finite subset $C_1, \dots, C_n$ of $\mathfrak{X}$, there are projections $P_1, \dots, P_n$ in $\mathfrak{Z}$ of sum 1 such that $\text{lub } C_i = \sum P_i C_i$. Using part (i) and the fact that $E(C) \leq E(D)$ for $C \leq D$ in $\mathfrak{Z}_h$, we have that

$$\text{lub } E(C_i) \leq E(\text{lub } C_i) = \sum P_i E(C_i) \leq \text{lub } E(C_i).$$

This means that

$$\begin{aligned}
E(\text{lub } \mathfrak{X}) &= E(\text{lub } \{C_i | \{C_i\} \text{ is a finite subset of } \mathfrak{X}\}) \\
&= \text{lub } \{E(\text{lub } C_i) | \{C_i\} \text{ is a finite subset of } \mathfrak{X}\} \\
&= \text{lub } \{E(C) | C \in \mathfrak{X}\}.
\end{aligned}$$

(iii) Let $C' = C + n^{-1}$. Then we have that

$$0 \leq (A - C)(E(C') - E(C)) \leq n^{-1}.$$

Thus we have

$$(A - C) \text{ glb } \{E(C') - E(C) | C' > C\} = 0.$$

(iv) Let C be in $\mathfrak{Z} - \text{Sp}_{\mathfrak{g}} A$ and let $\{P_n\}$ be a sequence of orthogonal projections in $\mathfrak{Z}$ of sum 1 such that $DP_n \geq n^{-1}P_n$ for every $n = 1, 2, \dots$. Because

$$P(E(C + D) - E(C - D)) \geq P(E(P(C + n^{-1})) - E(P(C - n^{-1})))$$

for every central projection P majorized by P_n and because CP is in $\mathfrak{Z}P - \text{Sp}_{\mathfrak{g}} P AP$ [12, 3.10], it is sufficient to show that the assumption that $(E(C + n^{-1}) - E(C - n^{-1})) = E$ is in $\mathfrak{g}$ leads to a contradiction. In fact, we have that $(C - A)(1 - E)$ is invertible in $\mathfrak{B}(1 - E)$ since $(C - A)^{\wedge}(\omega) \neq 0$ for every ω in Ω with $(1 - E)^{\wedge}(\omega) = 1$. If B is the inverse of $(C - A)(1 - E)$ in $\mathfrak{B}(1 - E)$, then B is the inverse of $C - A$ modulo $\mathfrak{g}$. This contradicts the definition of the essential central spectrum. Thus, we have that $c_{\mathfrak{g}}(E) = 0$.

Conversely, let C in $\mathfrak{Z}_h$ be such that $c_{\mathfrak{g}}(E(C + D) - E(C - D)) = 0$ for every D in $\mathfrak{Z}_h$ with $D > 0$. For every ζ in the spectrum of $\mathfrak{Z}$ and every $n = 1, 2, \dots$, the projection $E(C + n^{-1}) - E(C - n^{-1})$ is not in $\mathfrak{g} + [\zeta]$ [12, 3.1] and

$$\|(C^{\wedge}(\zeta) - A)(E(C + n^{-1}) - E(C - n^{-1}))(\mathfrak{g} + [\zeta])\| \leq n^{-1}.$$

Hence, each $C^{\wedge}(\zeta)$ is in the spectrum of the canonical image of A in $\mathfrak{Q}/(\mathfrak{g} + [\zeta])$. Q.E.D.

COROLLARY 2.6. *Let A be a normal element in a von Neumann algebra $\mathfrak{Q}$ with center $\mathfrak{Z}$ and let A_1 (resp. A_2) be $2^{-1}(A + A^*)$ (resp. $(2i)^{-1}(A - A^*)$). Then an element C in $\mathfrak{Z}$ is in the essential central spectrum of A modulo the central ideal $\mathfrak{g}$ if and only if*

$$c_{\mathfrak{g}}((E_{A_1}(C_1 + D_1) - E_{A_1}(C_1 - D_1))(E_{A_2}(C_2 + D_2) - E_{A_2}(C_2 - D_2))) = 0$$

for every $D_1, D_2 > 0$ in $\mathfrak{Z}$. Here

$$C_1 = 2^{-1}(C + C^*) \quad \text{and} \quad C_2 = (2i)^{-1}(C - C^*).$$

The next result is needed in §3.

COROLLARY 2.7. *If A is a selfadjoint operator in a type II_1 von Neumann algebra $\mathcal{Q}$, then there are two equivalent orthogonal projections of sum 1 that commute with A .*

PROOF. The set $\mathcal{X}$ of all selfadjoint elements C in the center $\mathcal{Z}$ of $\mathcal{Q}$ such that the value of the canonical operator-valued trace Φ of $\mathcal{Q}$ (cf. [7, III, 4, Definition 2]) does not exceed $\frac{1}{2}$ at the point $E(C) = E_A(C)$ is a bounded monotonely increasing net in $\mathcal{Z}$ (Proposition 2.5(i) and (ii)). We get an element $C_0 = \text{lub } \mathcal{X}$ such that $\Phi(E(C_0)) \leq \frac{1}{2}$ and such that $\Phi(E(C)) > \frac{1}{2}$ for every $C > C_0$ (Proposition 2.5(ii)). There is a projection E majorized by

$$F = \text{glb } \{E(C) - E(C_0) | C > C_0\}$$

such that

$$\Phi(E) = \frac{1}{2} - \Phi(E(C_0))$$

because $\mathcal{Q}$ is a continuous algebra. Since E commutes with A due to the fact $AF = C_0F$ (Proposition 2.5(iii)), the projections $E(C_0) + E$ and its orthogonal complement are equivalent orthogonal projections that commute with A . Q.E.D.

The next two lemmas are phrased in sufficient generality so that they will be applicable to the matrix decomposition in §3.

LEMMA 2.8. *Let E_i ($i = 1, 2$) be projections of central support 1 in the continuous von Neumann algebra $\mathcal{Q}$ and let ϕ be a σ -weakly continuous state of $\mathcal{Q}^\sim$ such that $\phi(E_1) \geq \alpha > 0$. Then for any β in the interval $]0, \alpha[$ and any A in $\mathcal{Q}$, there are projections F_i ($i = 1, 2$) of central support 1 majorized by the E_i respectively such that $\phi(F_1) \geq \beta$ and $F_2AF_1 = 0$.*

PROOF. Let $\mathcal{Z}$ be the center of $\mathcal{Q}$. Given any P in $(\mathcal{Z})$ it is sufficient to show that there is a Q in $(\mathcal{Z}P)$ and projections F_i of central support Q majorized by the respective E_i such that $\phi(F_1) \geq \beta Q$ and $F_2AF_1 = 0$. There is no loss of generality in the assumption that $P = 1$.

We may assume that E_2AE_1 is not 0. Let U be a partial isometry and B a positive operator in $\mathcal{Q}$ so that UB is the polar decomposition of E_2AE_1 . We may assume that the central support of the range projection F of B is 1 and that the central support P of

$$E = \text{glb } \{E_B(C') - E_B(C) | C' > C\},$$

where C is in $\mathcal{Z}F - \text{Sp } B$ in $\mathcal{Q}_F$, is either 0 or 1. On the one hand, if $P = 0$, then the strong limit of the monotonely decreasing sequence

$$G_n = E_B(C + n^{-1}) - E_B(C - n^{-1}) \quad (n = 1, 2, \dots)$$

is 0 and so we can find a number n and a Q in $(\mathfrak{L})$ such that

$$Q\phi(G_n) \leq (\alpha - \beta)Q,$$

and such that $FQ - G_n Q$ has central support Q . But the projection $G_n Q$ also has central support Q (Proposition 2.5(iv)). Hence the projections

$$F_1 = Q(E_1 - G_n) \quad \text{and} \quad F_2 = UG_n U^* Q$$

are projections of central support Q such that $\phi(F_1) \geq \beta Q$ and $F_2 A F_1 = 0$. On the other hand, if $P = 1$, then we may write E as the sum of an infinite sequence $\{G'_n\}$ of orthogonal projections of central support 1 which commutes with B (Proposition 2.5(iii)). There is a Q in $(\mathfrak{L})$ with $\phi(G'_n)Q \leq (\alpha - \beta)Q$. Setting $F_1 = (E_1 - G'_n)Q$ and $F_2 = UG'_n U^*$, we get two projections with $\phi(F_1) \geq \beta Q$ and $F_2 A F_1 = 0$. Q.E.D.

LEMMA 2.9. *For any finite subset $\mathfrak{X}$ of a properly infinite von Neumann algebra $\mathcal{Q}$ and any sequence $\{E_n\}$ of properly infinite projections of central support 1, there is a sequence $\{E'_n\}$ of equivalent projections of central support 1 such that $E'_m \leq E_m$ and $E'_n A E'_m = 0$ for every $1 \leq m < n < \infty$ and A in $\mathfrak{X}$.*

PROOF. We may assume that $\mathcal{Q}$ is either purely infinite or semifinite. First assume $\mathcal{Q}$ is purely infinite. By passing to a direct summand of $\mathcal{Q}$ and to subprojections of the E_n if necessary, we may assume that each E_n is σ -finite [7, III, 1, Lemma 7]. Since the least upper bound of the E_n is σ -finite, we may assume $\mathcal{Q}$ is σ -finite by reducing to $\mathcal{Q}_E$ if necessary. Applying Lemma 2.7 recursively, we may find projections

$$\{E_{nm} | m \geq n\} \quad (n = 1, 2, \dots)$$

in $\mathcal{Q}$ and corresponding maximal abelian projections $\{F_n\}$ in the commutant of the center $\mathfrak{Z}$ of $\mathcal{Q}$ such that

$$\begin{aligned} E_{nm} &\leq E_{np} \leq E_n && \text{for } n \leq p \leq m; \\ E_{nn} A E_{mn} &= 0 && \text{for } 1 \leq m < n \text{ and } A \text{ in } \mathfrak{X}; \end{aligned}$$

and

$$\tau_n(E_{n+n+k}) \geq 2 - \sum \{3^{-j} | 0 \leq j \leq k\} \text{ for } k = 0, 1, \dots$$

Here $\tau_n = \tau_{F_n}$. Setting $E'_n = \text{glb}_m E_{nm}$, we get that

$$\tau_n(E'_n) = \text{glb}_m \tau_n(E_{nm}) \geq 2^{-1}$$

so that E'_n is equivalent to 1 [7, III, 8, Theorem 1, Corollary 5] and that $E'_n A E'_m = 0$ for $1 \leq m < n < \infty$ and A in $\mathfrak{X}$.

Now let $\mathcal{Q}$ be semifinite. Suppose we have found equivalent finite projections $E'_1, E'_2, \dots, E'_n$ of central support 1 such that $E'_m \leq E_m$ for $m = 1, 2, \dots, n$ and $E'_p A E'_m = 0$ for $1 \leq m < p \leq n$ and A in $\mathcal{X}$. The least upper bound F of the range projections of $E_{n+1} A E'_m$ for $1 \leq m \leq n$ and A in $\mathcal{X}$ is a finite projection, and so there is a projection E'_{n+1} with $E'_n \sim E'_{n+1} \leq E_{n+1} - F$. By induction the required sequence $\{E'_n\}$ exists. Q.E.D.

We now obtain the desired extension of Theorem 2.2.

THEOREM 2.10. *Let $\mathcal{Q}$ be a von Neumann algebra with center $\mathcal{Z}$, and let $\mathcal{G} = \mathcal{G}_{\mathcal{Q}}(G)$ be a central ideal of $\mathcal{Q}$ in canonical form. Suppose that $\mathcal{Q}(1 - Q)$ is a continuous von Neumann algebra. Let 0 be in the essential central spectrum modulo $\mathcal{G}$ of an element A in $\mathcal{Q}$. For any sequence $\{\alpha_n\}$ of positive numbers and any finite subset $\mathcal{X}$ of $\mathcal{Q}$, there is a sequence $\{E_n\}$ of projections in $\mathcal{Q}$ such that*

- (1) $\dim E_n > \dim \mathcal{G}$;
- (2) $E_n B E_m = 0$ for all $1 \leq m < n < \infty$ and B in $\mathcal{X}$; and
- (3) $\|E_n A E_n\| \leq \alpha_n$ for all n .

In particular, if 1 is in $\mathcal{X}$, then the projections $\{E_n\}$ are orthogonal.

REMARK. For the applications in §3 it is important to notice that the strong radical $\mathcal{G}$ of a properly infinite algebra $\mathcal{Q}$ satisfies the hypothesis of Theorem 2.10. We recall, in fact, that $\mathcal{Q}Q$ is the weak closure of the central ideal $\mathcal{G}_{\mathcal{Q}}(G)$ (cf. Introduction) and so $\mathcal{Q}(1 - Q)$ is of type III or (0) if $\mathcal{G}_{\mathcal{Q}}(G) = \mathcal{G}$ because $\mathcal{G}$ contains all finite projections.

If $\mathcal{Q}$ has a nonzero type I direct summand $\mathcal{Q}P$ with $\mathcal{G}P = (0)$ and E is an abelian projection of central support P , there is no orthogonal sequence $\{E_n\}$ of projections of central support P that satisfies (3) for $\alpha_n = 2^{-1}$ ($n = 1, 2, \dots$) and $A = 1 - E$.

PROOF. We may assume that $P_{\mathcal{G}} = 1$, and by passing to direct summands, that $\mathcal{Q}$ is continuous and $\mathcal{G} = (0)$ or that the support of $\mathcal{G}$ is 1.

If $\mathcal{Q}$ is continuous and $\mathcal{G} = (0)$, then there is a sequence $\{F_n\}$ of projections with central support 1 such that $\|F_n A F_n\| \leq \alpha_n$. The desired sequence $\{E_n\}$ may be constructed in a recursive fashion from Lemma 2.8 (cf. Proof, Lemma 2.9).

Now suppose the support of $\mathcal{G}$ is 1. We also suppose that $\mathcal{X}$ contains the identity 1. There is a sequence $\{F_n\}$ of projections in $\mathcal{Q}$ with $\dim F_n > \dim \mathcal{G}$ and $\|F_n A F_n\| \leq \alpha_n$ for every $n = 1, 2, \dots$ (Theorem 2.2). The projections F_n must be properly infinite projections of central support 1 since every finite projection is in $\mathcal{G}$ (12, 2.2). Let $\mathcal{S}$ be the set of all families

$$S = \{E_{ni} | (n, i) \in N \times I\}$$

of nonzero projections in $\mathcal{Q}$ with the following properties

- (i) $E_{mi} \sim E_{ni}$ for all m, n in N and $i \in I$;

(ii) $E_{nj}BE_{mi} = 0$ for all distinct pairs $(m, i), (n, j)$ in $N \times I$ and B in $\mathfrak{U} = \mathfrak{X} \cup \mathfrak{X}^* \cup \{A, A^*\}$; and

(iii) $\|E_{ni}AE_{ni}\| \leq \alpha_n$ for n in N and $i \in I$.

Here $\mathfrak{X}^* = \{B^* | B \in \mathfrak{X}\}$ and N is the set of natural numbers. The set $\mathfrak{S}$ is nonvoid, and in fact, for every nonzero central projection P , there is a sequence $\{E'_n\}$ of equivalent orthogonal projections of central support P such that $E'_n \leq F_n$ and $E'_nBE'_m = 0$ for all $1 \leq m < n < \infty$ and B in $\mathfrak{U}$ (Lemma 2.9). By Zorn's lemma we may find a maximal set $S = \{E_{ni} | (n, i) \in N \times I\}$ in the partial ordering on $\mathfrak{S}$ given by the relation

$$(3) \quad \{E'_{ni} | (n, i) \in N \times I'\} < \{E''_{ni} | (n, i) \in N \times I''\}$$

if and only if $I' \subset I''$ and $E'_{ni} = E''_{ni}$ for $i \in I'$. For convenience, let k be a point not in I . Let $\{N_m | m \in N\}$ be a partition of the natural numbers into countably many disjoint infinite sets such that the smallest number in the set N_m is not smaller than m , and let

$$E_m = \sum \{E_{ni} | (n, i) \in N_m \times I\}$$

for $m = 1, 2, \dots$. We show that each projection E_n has dimension greater than that of $\mathfrak{g}$ by showing $c_{\mathfrak{g}}(E_n) = 0$. First we notice that each projection $E'_n = \sum \{E_{ni} | i \in I\}$ has central support P equal to 1; otherwise, there would be a sequence $\{E_{nk}\}$ of central support $1 - P$ such that

$$\{E_{ni} | (n, i) \in N \times (I \cup \{k\})\}$$

lies in $\mathfrak{S}$. Since the projections E'_n are equivalent, the projections E_n are equivalent, properly infinite, orthogonal projections of central support 1 [7, III 2, Proposition 10]. We show that $c_{\mathfrak{g}}(E_1) = P$ is zero. On the contrary, assume P is nonzero. We may write E_1 as the sum of an infinite sequence $\{G_n\}$ of orthogonal projections each equivalent to E_1 and thus to each E_m . Because

$$P \sum E_n < P \sum G_n = PE_1,$$

we get that $P \sum E_n$ is in $\mathfrak{g}$. The least upper bound E of the range supports of the finite set of elements $\{PB \sum E_n | B \in \mathfrak{U}\}$ is also a projection in $\mathfrak{g}$. This means that the algebra $\mathcal{C} = \mathcal{A}_{(P-E)}$ is properly infinite and that 0 is in the essential range of the element $C = (P - E)A(P - E)$ in $\mathcal{C}$ modulo the central ideal $\mathfrak{g}' = \mathfrak{g} \cap \mathcal{C}$ of $\mathcal{C}$. Indeed, if ϕ is a positive $\mathfrak{X}$ -module homomorphism of the algebra $\mathcal{A}$ into $\mathfrak{X}$ with $\phi(1) = 1$, $\phi(A) = 0$ and $\phi(\mathfrak{g}) = (0)$, then the function ψ on $\mathcal{C}$ given by

$$\psi(D) = (P - E)\phi(D)$$

is a positive module homomorphism of $\mathcal{C}$ into its center $\mathfrak{Z}(P - E)$ with $\psi(P - E) = P - E$, $\psi(C) = 0$, and $\psi(\mathfrak{G}') = (0)$. There is a sequence $\{F'_n\}$ of projections in $\mathcal{C}$ with $\dim F'_n > \dim \mathfrak{G}'$ such that $\|F'_n A F'_n\| \leq \alpha_n$. Since the weak closure of $\mathfrak{G}'$ is $\mathcal{C}$ and since the orthogonal complement of the largest central projection in $\mathcal{C}$ that is in $\mathfrak{G}'$ is 0, the projections F'_n are properly infinite and of central support $P - E$ in $\mathcal{C}$. There are equivalent projections $\{E_{nk} | n \in N\}$ of central support 1 such that $E_{mk} \leq F'_m$ and $E_{nk} B E_{mk} = 0$ for $1 \leq m < n < \infty$ and B in the set $\{(P - E)D(P - E) | D \in \mathfrak{G}\}$. Then the set

$$\{E_{ni} | (n, i) \in N \times (I \cup \{k\})\}$$

is in $\mathfrak{S}$. This is a contradiction. Hence, we have that $c_{\mathfrak{G}}(E_1) = 0$ and that $\dim E_n > \dim \mathfrak{G}$. Thus $\{E_n\}$ is a sequence of projections satisfying property (1). We show that $\{E_n\}$ also satisfies (2) and (3). In fact, for $1 \leq m < n < \infty$ and B in $\mathfrak{X}$, we have that

$$\|E_n B E_m\| = \left\| \sum \{E_{pi} B E_{qj} | (p, q, i, j) \in N_n \times N_m \times I \times I\} \right\| = 0$$

from (ii), and that

$$\begin{aligned} \|E_m A E_m\| &= \left\| \sum \{E_{pi} A E_{qj} | (p, q, i, j) \in N_m \times N_m \times I \times I\} \right\| \\ &= \left\| \sum \{E_{pi} A E_{pi} | (p, i) \in N_m \times I\} \right\| \\ &= \text{lub } \{\|E_{pi} A E_{pi}\| | (p, i) \in N_m \times I\} \leq \alpha_m \end{aligned}$$

from (ii) and (iii). Q.E.D.

We now examine the relation of the essential central spectrum and essential central range.

COROLLARY 2.11. *Let $\mathcal{Q}$ be a von Neumann algebra and let $\mathfrak{G} = \mathfrak{G}_{\mathcal{Q}}(G)$ be a central ideal of $\mathcal{Q}$ in canonical form. Suppose that $\mathcal{Q}(1 - Q)$ is a continuous von Neumann algebra. Then an element C in $\mathfrak{Z}$ is in the essential central range of an element A in $\mathcal{Q}$ modulo the central ideal $\mathfrak{G}$ if and only if there is a projection E in $\mathcal{Q}$ with $\dim E > \dim \mathfrak{G}$ such that CE is in the essential central spectrum of EAE in $\mathcal{Q}_E$ modulo the central ideal $\mathfrak{G} \cap E\mathcal{Q}E$.*

PROOF. There is no loss of generality in that assumption that $P_{\mathfrak{G}} = 1$ and $C = 0$. If 0 is in $\mathfrak{K}_{\mathfrak{G}}(A)$, there is a sequence $\{E_n\}$ of projections in $\mathcal{Q}$ with $\dim E_n > \dim \mathfrak{G}$, $\|E_n A E_n\| \leq m^{-1}$, and $E_n B E_m = 0$ for $1 \leq m < n < \infty$ and B in the set $\{1, A, A^*\}$ (Theorem 2.10). Let E be the projection $E = \sum E_n$. Each E_m has dimension greater than that of $\mathfrak{G}' = \mathfrak{G} \cap E\mathcal{Q}E$ since

$$(4) \quad c_{\mathcal{G}}(E_m) = c_{\mathcal{G}}(E_m)E = 0.$$

Since

$$\lim \|EAE E_m\| = \lim \|E_m A E_m\| = 0,$$

we get that 0 is in $\mathfrak{Z}E - \text{Sp}_{\mathcal{G}} EAE$.

Conversely, let E be a projection in $\mathcal{Q}$ with $\dim E > \dim \mathcal{G}$ such that 0 is in the essential central spectrum of EAE modulo $\mathcal{G}' = \mathcal{G} \cap \mathcal{Q}_E$. For every $m = 1, 2, \dots$, there is an E_m in $\mathcal{Q}_E$ with $\dim E_m > \dim \mathcal{G}'$ such that $\|E_m EAE E_m\| \leq m^{-1}$ (cf. [12, 3.16]). Because E has central support 1, we get that $\dim E_m > \dim \mathcal{G}$ from relation (4). We conclude from Theorem 2.2 that $\mathcal{K}_{\mathcal{G}}(A)$ contains 0. Q.E.D.

3. Selfadjoint commutators. In this section the selfadjoint commutators in properly infinite von Neumann algebras are characterized as those elements that contain 0 in their essential central range modulo the strong radical. A matrix form similar to that used for bounded operators on a separable space in [1] is found for such operators.

DEFINITION 3.1. A sequence $\{E_n | 1 \leq n < m\}$ (where m is a finite or countable cardinal) of orthogonal projections in a properly infinite von Neumann algebra is said to be a *matrix base* if $1 \sim E_1 \sim E_2 \sim \dots$ and $\sum E_n = 1$.

Let $\{E_n | 1 \leq n \leq m\}$ be a matrix base for the properly infinite algebra $\mathcal{Q}$ and let U_n be partial isometries of $\mathcal{Q}$ with domain support E_n and range support 1. Then each element A in $\mathcal{Q}$ can be written as

$$A = \sum E_n A E_m = \sum U_n^* \cdot U_n A U_m^* \cdot U_m.$$

The $m \times m$ -matrix $(U_n A U_m^*)$ is said to be a *matrix with regard to the base* $\{E_n\}$. The matrix depends on the choice of the partial isometries; however, in the sequel the choice of the matrix base is critical and the choice of the partial isometries is immaterial.

We now begin a series of matrix reductions. The first extends the result in Corollary 2.7.

LEMMA 3.2. *For every selfadjoint operator A in a properly infinite von Neumann algebra $\mathcal{Q}$, there is a matrix base $\{E_i | 1 \leq i \leq 2\}$ for $\mathcal{Q}$ that commutes with A .*

PROOF. Let $\mathfrak{J}$ be the strong radical of $\mathcal{Q}$, let $\mathfrak{Z}$ be the center of $\mathcal{Q}$, and let C_u and C_l be the least upper bound and greatest lower bound respectively of $\mathfrak{Z} - \text{Sp}_{\mathcal{G}} A$ [12, 3.11]. Since there is a projection P in $\mathfrak{Z}$ with $PC_u > PC_l$ and $(1 - P)C_u = (1 - P)C_l$, we may assume that either $C_u > C_l$ or that $C_u = C_l$.

If $C_u > C_l$, then the projections $E = E_A(C_u + D) - E_A(C_u - D)$ and $1 - E$ are equivalent to 1 whenever $0 < D \leq 2^{-1}(C_u - C_l)$ due to Proposition 2.5(iv). Hence, we may assume $C_u = C = C_l$.

Let E denote the range projection of the selfadjoint operator $C - A$ in $\mathfrak{F}$; then the projection $c_{\mathfrak{F}}(E)(1 - E)$, which annihilates $C - A$, is equivalent to $c(E) = c_{\mathfrak{F}}(E)$ and can be written as the sum of two orthogonal projections E_1, E_2 both equivalent to $c(E)$ [7, III, 8, Theorem 1, Corollary 2]. The projections $E_1 + c(E)E$ and E_2 form a matrix base for $\mathcal{A}(E)$ that commute with $c(E)(C - A)$ and thus with $c(E)A$. So we may assume that $c(E) = 0$, or equivalently, that $E \sim 1$.

It is sufficient to show that given any selfadjoint operator A in $\mathfrak{F}$ with range support equivalent to 1, there are equivalent orthogonal projections of sum 1 that commute with A . We prove first that there are at least two nonzero equivalent orthogonal projections majorized by the range support E of A that commute with A . Assume for the present that $E = 1$. Let C_1, C_2 be in $\mathfrak{X}$ with $C_1 > C_2 > 0$ and let $F(C_1, C_2)$ be the projection in $\mathfrak{F}$ given by

$$F(C_1, C_2) = (E(C_1) - E(C_2)) + (E(-C_2) - E(-C_1))$$

where $E(C_2) = E_A(C_2)$ (cf. Definition 2.4). By decomposing into the product of two algebras, we may suppose that either there is a C_1 in $\mathfrak{X}$ with

$$\|A\| + 1 = C_0 > C_1 > 0$$

such that $F(C_0, C_1)$ is properly infinite of central support 1 or that $F(C_0, C_1)$ is finite or zero for every $C_0 > C_1 > 0$ (Proposition 2.5(i)).

First assume $F(C_0, C_1)$ is properly infinite of central support 1. We show that there is a C_2 in $\mathfrak{X}$ with $C_1 > C_2 > 0$ such that

$$F(C_1, C_2) \succ F(C_0, C_1).$$

In fact, we show more: for any C, C' in $\mathfrak{X}$ with $C > C' > 0$ so that $F(C, C')$ is properly infinite of central support 1, there is a C'' in $\mathfrak{X}$ with $C' > C'' > 0$ such that

$$F(C', C'') \succ F(C, C').$$

Since we have that

$$PE(D) = PE(PD)$$

for every selfadjoint D in $\mathfrak{X}$ (Proposition 2.5(i)), it is sufficient to show that for any P in $(\mathfrak{X})$ there is a Q in $(\mathfrak{X}P)$ and a C'' in ZQ with $C'Q > C''Q > 0$ and

$$QF(C'Q, C''Q) \succ QF(CQ, C'Q).$$

There is no loss of generality in the assumption that $P = 1$. We show that the relation

$$F(C, C') > F(C', C'')$$

for all C'' with $C' > C'' > 0$ leads to a contradiction. In fact let $\{F_n\}$ be a sequence of orthogonal projections of sum $F(C, C')$ such that $F_n \sim F(C, C')$ for every n [7, III, 8, Theorem 1, Corollary 2]. We have that

$$1 \sim 1 - F(C_0, C') = \sum F(m^{-1}C', (m+1)^{-1}C') < \sum F_n \leq F(C, C')$$

because the range projection of A given by $1 - E(0+) + E(0)$ is equal to 1. This is a contradiction due to the fact $F(C, C')$ is in $\mathcal{J}$. So there is a C'' in $\mathcal{X}$ with $C' > C'' > 0$ such that

$$F(C', C'') > F(C, C').$$

Noting that $F(C', C'')$ in the previous expression is properly infinite of central support 1, we may find a strictly decreasing sequence $\{C_n\}$ in $\mathcal{X}^+$ such that $F(C_0, C_1) < F(C_1, C_2) < \dots$. Setting

$$E_1 = \sum F(C_{2n}, C_{2n+1}) \quad \text{and} \quad E_2 = \sum F(C_{2n-1}, C_{2n}),$$

we obtain two orthogonal projections E_1 and E_2 of sum 1 such that

$$E_2 < E_1 < F(C_0, C_1) + E_2.$$

Since $F(C_0, C_1)$ is in the strong radical, we get that $E_1 \sim E_2 \sim 1$.

Now we assume that $F(C_0, n^{-1})$ is a finite or zero projection for every $n = 1, 2, \dots$. If $\mathcal{A}$ is a continuous algebra, then there are, for each $n = 1, 2, \dots$, two equivalent orthogonal projections of sum $F(C_0, n^{-1})$ which commute with $AF(C_0, n^{-1})$ and consequently with A (Corollary 2.7). Since $F(C_0, n^{-1})$ is nonzero for some n (Proposition 2.5(iii)), we obtain nonzero equivalent orthogonal projections commuting with A . Therefore, we may assume that $\mathcal{A}$ is of type I. There is an n such that $F(C_0, n^{-1})$ is not an abelian projection; otherwise, the properly infinite projection 1 can be expressed as the least upper bound of a monotonely increasing sequence $\{F(C_0, n^{-1})\}$ of abelian projections which is also an abelian projection. This means that there is an n and a Q in $(\mathcal{X})$ such that $QF(C_0, n^{-1})$ is the sum of m ($m \geq 2$) orthogonal abelian projections of central support Q . Because $AF(C_0, n^{-1})Q$ can be represented as a diagonal $m \times m$ operator matrix over $\mathcal{X}F(C_0, n^{-1})Q$ based on matrix units arising from the m equivalent orthogonal abelian projections of sum $F(C_0, n^{-1})Q$ [6], we may find two nonzero orthogonal equivalent projections (actually abelian projections) which commute with $AF(C_0, n^{-1})Q$ and thus with A .

In every case we have found two nonzero equivalent orthogonal projections that commute with A . We now remove the assumption that the range projection E is equal to 1. Let V be an isometry in $\mathcal{Q}$ whose range support is E . The element V^*AV is a selfadjoint element in the strong radical whose range support is 1. As was just shown, there are nonzero orthogonal equivalent projections E_1 and E_2 that commute with VAV^* . The projections VE_1V^* and VE_2V^* are nonzero equivalent orthogonal projections majorized by E that commute with A .

We now construct the desired projections by a maximality argument. Let $\{F_i, G_i | i \in I\}$ be a maximal set of nonzero mutually orthogonal projections majorized by E that commute with A such that $F_i \sim G_i$ for every i . The projections $F = \sum F_i$ and $G = \sum G_i$ are equivalent orthogonal projections majorized by E that commute with A . The projection

$$E' = c(F)(E - (F + G))$$

is equivalent to $c(F)$ since $c(F) = c(G)$, and the range support E' of AE' in the strong radical of the properly infinite von Neumann algebra $\mathcal{Q}c(F)$ is equivalent to $c(F)$. This means $c(F) = 0$; otherwise, the arguments of the preceding paragraphs can be applied to show that $\{F_i, G_i\}$ is not maximal. Thus we have that the projections $F, 1 - F$ form a matrix base that commutes with A . Q.E.D.

REMARK. Combining the results of Corollary 2.7, Lemma 3.2, and [6], we obtain that, for every selfadjoint operator A in a von Neumann algebra with no type I_n (n odd) part, there are two equivalent orthogonal projections of sum 1 commuting with A .

The next lemma extends the results of Anderson and Stampfli [2].

LEMMA 3.3. *Let A be an element in a properly infinite von Neumann algebra $\mathcal{Q}$; then there is a matrix base E_1, E_2 (resp. F_1, F_2) of $\mathcal{Q}$ such that $F_1AE_2 = F_2AE_1 = 0$.*

PROOF. Let B be a positive selfadjoint operator in $\mathcal{Q}$ and let U be a partial isometry of $\mathcal{Q}$ so that $A = UB$ is the polar decomposition of A . The projections $c(U^*U)(1 - U^*U)$ and $c(U^*U)(1 - UU^*)$ may be written as the sum of two equivalent orthogonal projections E, E' and F, F' respectively. Here $c(U^*U) = c_{\mathcal{Q}}(U^*U)$ where $\mathcal{Q}$ is the strong radical of $\mathcal{Q}$. The projections

$$E_1 = E + c(U^*U)U^*U, \quad E_2 = E'$$

and

$$F_1 = F + c(U^*U)UU^*, \quad F_2 = F'$$

are equivalent orthogonal projections of sum $c(U^*U)$ respectively with

$$F_1 A E_2 = F_2 A E_1 = 0.$$

So we may assume that $c(U^*U) = 0$. There are equivalent orthogonal projections E, E' of sum U^*U that commute with B due to the fact that U^*U is properly infinite (Lemma 3.2). The projections UEU^* and $UE'U^*$ are orthogonal equivalent projections of sum UU^* . Setting

$$E_1 = E + (1 - U^*U), \quad E_2 = E'$$

and

$$F_1 = UEU^* + (1 - UU^*), \quad F_2 = UE'U^*,$$

we get matrix bases that satisfy the desired relations. Q.E.D.

We now are ready to characterize a selfadjoint commutator. We treat the product of σ -finite type I algebras separately. This case is slightly anomalous and in fact follows from the preceding lemmas via the analysis of J. Anderson [1]. In the sequel let $\alpha_n = ((n+1)!)^{-1}$ for $n = 1, 2, \dots$

THEOREM 3.4. *Let $\mathcal{Q}$ be the product of σ -finite properly infinite type I von Neumann algebras. Then an element A in $\mathcal{Q}$ is a selfadjoint commutator in $\mathcal{Q}$ if and only if 0 is in the essential central range of A modulo the strong radical $\mathcal{J}$ of $\mathcal{Q}$.*

PROOF. First let A have the matrix representation

$$(5) \quad \begin{pmatrix} * & 0 \\ 0 & B \end{pmatrix},$$

where $B \in \mathcal{J}$. Since $\mathcal{K}(B^*B + BB^*) = \{0\}$, we may write B as the infinite matrix operator $B = (B_{ij})$ over $\mathcal{Q}$ with $\|B_{ij}\| \leq 2\|B\| \min\{\alpha_i, \alpha_j\}$. In fact, by applying Theorem 2.10 to $B^*B + BB^*$, we may find a sequence $\{E_n\}$ of orthogonal projections in $\mathcal{Q}$ such that $\dim E_n > \dim \mathcal{J}$ and

$$\|E_n(B^*B + BB^*)E_n\| \leq 4\alpha_n^2\|B\|^2$$

for all $1 \leq n < \infty$. Replacing E_1 by $1 - \sum \{E_n | n \geq 2\}$, we get a matrix base $\{E_n\}$ of $\mathcal{Q}$ such that

$$\|E_n B E_m\| \leq 2\|B\| \min\{\alpha_m, \alpha_n\}.$$

The calculation of M. David [5, Theorem 3] now shows that A is a selfadjoint commutator.

Now let A be an operator in $\mathcal{Q}$ such that $\mathcal{K}(A)$ contains 0. Applying Theorem 2.10, we may find an orthogonal sequence $\{F_n\}$ of maximal abelian projections of $\mathcal{Q}$ of sum F such that $1 - F \sim F \sim 1$ and such that

$$\|F_m A F_n\| \leq n^{-1} \delta_{mn}$$

for all $m, n = 1, 2, \dots$ [7, III, 8, Theorem 1, Corollary 5]. Here δ is the Kronecker delta. We have that FAF is in $\mathcal{J}$ since $G = \sum \{F_n | 1 \leq n \leq m\}$ is in $\mathcal{J}$ and

$$\|FAF(1 - G)\| = \text{lub} \{\|F_n A F_n\| | n > m\} \leq (m + 1)^{-1}.$$

The 2,2 term of the matrix of A induced by the matrix base $1 - F$, F is in $\mathcal{J}$. Using Lemma 3.3, we can follow the analysis of J. Anderson [1] to write A as an 8×8 -matrix over $\mathcal{Q}$ such that each of the four 2×2 diagonal blocks is of the form (5). Thus A is a selfadjoint commutator.

Conversely, let $A = BC - CB$ with $B = B^*$ and C in $\mathcal{Q}$. We may assume that 0 is in $\mathfrak{X} - \text{Sp}_{\mathcal{J}} B$. Given $\varepsilon > 0$, there is a projection E in $\mathcal{Q}$ with $\dim E > \dim \mathcal{J}$ and with $\|BE\| = \|EB\| < \varepsilon$ (Proposition 2.5(iv)). We have that $\|EAE\| \leq 2\varepsilon\|C\|$, and thus that $0 \in \mathcal{K}_{\mathcal{J}}(A)$. Q.E.D.

Now we can restrict our attention to properly infinite algebras with no σ -finite type I direct summands.

LEMMA 3.5. *Let A be a selfadjoint operator in a properly infinite von Neumann algebra $\mathcal{Q}$ with no σ -finite type I direct summands and let $\{E_n\}$ be a sequence of projections in $\mathcal{Q}$ which commute with A ; then for every $n = 1, 2, \dots$, there are equivalent orthogonal projections E_{n1} and E_{n2} commuting with A and majorized by E_n such that $\text{lub} (E_n - (E_{n1} + E_{n2}))$ is in the strong radical of $\mathcal{Q}$.*

PROOF. We may assume that either $\mathcal{Q}$ is a continuous σ -finite algebra or that $\mathcal{Q}$ has no σ -finite direct summand. In case $\mathcal{Q}$ is a continuous σ -finite algebra, we show that E_n may in fact be written as the sum of two equivalent orthogonal projections commuting with A . Let P_n be the largest central projection in $\mathcal{Q}$ such that $E_n P_n$ is finite. Then $E_n P_n$ (resp. $E_n(1 - P_n)$) may be written as the sum of two equivalent orthogonal projections commuting with $A E_n P_n$ (resp. $A E_n(1 - P_n)$) due to Corollary 2.7 (resp. Lemma 3.2). So E_n may be written as the sum of two equivalent orthogonal projections commuting with A .

Now suppose $\mathcal{Q}$ has no σ -finite direct summands. For every $n = 1, 2, \dots$, we may use the reasoning of the preceding paragraph to find two equivalent orthogonal projections E_{n1}, E_{n2} commuting with A and majorized by E_n so that $E_n - (E_{n1} + E_{n2})$ is a finite projection. To complete the proof it is sufficient to show that the least upper bound F of a monotonely increasing sequence $\{F_n\}$ of finite projections in a von Neumann algebra with no σ -finite direct summands is in the strong radical [7, III, 2, Proposition 5]. Since the strong radical is a central ideal, and since any direct summand of a finite projection is finite, it is sufficient to show that there is a nonzero central projection Q such that FQ is in the strong radical. Let x be a unit vector in

the Hilbert space of $\mathcal{Q}$. For each $n = 1, 2, \dots$, there is an orthogonal set $\{Q_i\}$ central projections of sum 1 such that $F_n Q_i$ is σ -finite [7, I, 6, Proposition 9(iii)]. There is a finite subset of the Q_i of sum Q_n such that $\|x - Q_n x\| \leq 2^{-n-1}$. Setting Q equal to the greatest lower bound of the monotonely decreasing sequence $\{Q_1 Q_2 \cdots Q_n\}$ of central projections, we obtain the required nonzero projection because $\|x - Qx\| \leq 2^{-1}$ and because FQ is σ -finite and consequently in the strong radical. Q.E.D.

If 0 is in the essential central range of A modulo the strong radical of the properly infinite algebra $\mathcal{Q}$, the projections $\{E_n\}$ of Theorem 2.10 induce the matrix decomposition $A = (C_{ij})$ over $\mathcal{Q}$ with $\|C_{mn}\| \leq \alpha_n \delta_{mn}$ for $m, n \geq 2$. Here we replace E_1 of Theorem 2.10 by $1 - \sum \{E_n | n \geq 2\}$, which majorizes E_1 and is thus equivalent to 1. We may also choose the E_n so that $E_m B E_n = 0$ for $2 \leq m < n < \infty$ and B in a preassigned finite subset of $\mathcal{Q}$. We now obtain a second matrix form for A .

LEMMA 3.6. *Let A be an element in a properly infinite von Neumann algebra $\mathcal{Q}$ with no σ -finite type I direct summands and let $\{E_n\}$ be an infinite matrix base of $\mathcal{Q}$ such that $\|E_n B E_m\| = 0$ for $2 \leq m < n < \infty$ and B in the set $\mathcal{X} = \{A^*, A, A^* A, A A^*\}$. Then there are orthogonal equivalent projections E_{n1} and E_{n2} of sum E_n such that*

$$E_{11} A \sum \{E_{n2} | n \geq 2\} = E_{12} A \sum \{E_{n1} | n \geq 2\} = 0.$$

PROOF. Let U be a partial isometry and B a positive element of $\mathcal{Q}$ so that UB is the polar decomposition of $E_1 A \sum \{E_n | n \geq 2\}$. Because each E_n ($n \geq 2$) commutes with B due to the validity of the relation

$$E_m B^* B E_n = E_m A^* (E_1 + E_m) (E_n + E_1) A E_n = E_m A^* A E_n = 0$$

for $2 \leq m < n < \infty$, each E_n commutes with the range projection E of B . There are equivalent orthogonal projections F_{n1} and F_{n2} ($n \geq 2$) commuting with B and majorized by $E E_n$ such that

$$F = \sum \{E E_n - (F_{n1} + F_{n2}) | n \geq 2\}$$

is in the strong radical $\mathcal{J}$ of $\mathcal{Q}$ (Lemma 3.5). The projection $E_n - E_n E$ ($n \geq 2$) (resp. $E_1 - U U^*$) is also the sum of orthogonal projections G_{n1}, G_{n2}, G_{n3} such that $G_{n1} \sim G_{n2}$ and G_{n3} is in $\mathcal{J}$. Then we let

$$E_{n1} = (E E_n - F_{n2}) + G_{n1} + G_{n3}$$

and

$$E_{n2} = F_{n2} + G_{n2}$$

for $n \geq 2$ and

$$E_{11} = U \sum \{EE_n - F_{n2} | n \geq 2\} U^* + G_{11} + G_{13}$$

and

$$E_{12} = U \sum \{F_{n2} | n \geq 2\} U^* + G_{12}.$$

We have that $c(E_{n1}) = c(E_{n2}) = 0$ for $n \geq 2$ and so $E_{n1} \sim 1 \sim E_{n2}$. Here $c = c_g$. Also we have that

$$\begin{aligned} c(E_{11}) &= c\left(U\left(F + \sum \{F_{n1} | n \geq 2\}\right)U^* + G_{11} + G_{13}\right) \\ &= c\left(U\left(\sum \{F_{n1} | n \geq 2\}\right)U^* + G_{11}\right) \\ &= c\left(U\left(\sum \{F_{n2} | n \geq 2\}\right)U^* + G_{12}\right) = c(E_{12}), \end{aligned}$$

and thus $E_{11} \sim 1 \sim E_{12}$. We also have that

$$E_{11} A \sum \{E_{n2} | n \geq 2\} = U\left(E - \sum \{F_{n2} | n \geq 2\}\right) \sum \{F_{n2} | n \geq 2\} B = 0$$

and

$$E_{12} A \sum \{E_{n1} | n \geq 2\} = U \sum \{F_{n2} | n \geq 2\} \left(E - \sum \{F_{n2} | n \geq 2\}\right) B = 0.$$

Q.E.D.

Now if 0 is in the essential central spectrum of A in the properly infinite von Neumann algebra $\mathcal{Q}$, then we may employ Theorem 2.10 and Lemma 3.6 (as applied to A^*) to write A as a 4×4 -matrix (A_{ij}) over $\mathcal{Q}$ such that $A_{41} = A_{32} = 0$ and such that A_{33} and A_{44} can be further decomposed to infinite operator matrices (C_{ij}) over $\mathcal{Q}$ of the form $\|C_{ij}\| \leq \alpha_i \delta_{ij}$. To apply Lemma 3.6 to this new matrix we need the next lemma.

LEMMA 3.7. *Let A be an element of a properly infinite von Neumann algebra $\mathcal{Q}$ and let $\{F_n\}$ be an infinite matrix base for $\mathcal{Q}$ such that $F_1 A F_m = 0$ for $m \geq 2$ and $\|F_n A F_m\| \leq \alpha_n \delta_{mn}$ for $m, n \geq 2$; then there is an infinite matrix base $\{E_n\}$ for $\mathcal{Q}$ such that $E_n B E_m = 0$ for $2 \leq m < n < \infty$ and B in the set $\mathcal{X} = \{1, A, A^*, A A^*, A^* A\}$, and $\|E_1 A E_m\| \leq \alpha_m$, $\|E_m A E_m\| \leq \alpha_m$ for $m \geq 2$.*

PROOF. We use the same arguments as in the proof of Theorem 2.10. Let $\mathcal{S}$ be the family of all sets

$$S = \{E_{ni} | (n, i) \in N \times I\}$$

of nonzero projections in $\mathcal{Q}$ such that

- (i) $E_{mi} \sim E_{ni}$ for all m, n in N and $i \in I$;
- (ii) $E_{nj}BE_{mi} = 0$ for all B in $\mathfrak{X}$ and distinct pairs $(m, i), (n, j)$ in $N \times I$; and
- (iii) $E_{ni} \leq F_n$ for all n in N and i in I .

From Lemma 2.9, we conclude that $\mathfrak{S}$ is nonvoid, and from Zorn's lemma we find a maximal set $S = \{E_{ni} | (n, i) \in N \times I\}$ in $\mathfrak{S}$ in the partial ordering (3). Let $\{N_n | n \in N\}$ be a partition of N into a sequence of infinite sets such that the smallest element of N_n is not less than n . As in Lemma 2.9, we can show that each of the projections

$$E_n = \sum \{E_{mi} | (m, i) \in N_n \times I\}$$

is a properly infinite projection of central support 1. We show that each E_n is equivalent to 1. If $\mathcal{Q}$ is the product of σ -finite algebras, then each E_n is equivalent to 1 since each E_n is properly infinite. So we may assume that $\mathcal{Q}$ has no σ -finite direct summands. We show that $E_n \sim 1$ by showing that the largest central projection P such that PE_n is in the strong radical $\mathfrak{J}$ is $P = 0$. We argue by contradiction. As in Theorem 2.10, we have that $P \sum E_n$ is in $\mathfrak{J}$. For every $n = 1, 2, \dots$, the orthogonal complement G_n in PF_n of the least upper bound of the range supports of the elements $F_nBP \sum E_n$ for B in $\mathfrak{X}$ is equivalent to P due to the fact this least upper bound is in $\mathfrak{J}$. Setting $G = \sum G_n$ and applying Lemma 2.9 to the element GAG in the properly infinite algebra $\mathcal{Q}_G$ and to the sequence $\{G_n\}$ of projections, we may find a sequence $\{G_{nk}\}$ of projections in $\mathcal{Q}_G$ so that

$$\{E_{ni} | (n, i) \in N \times (I \cup \{k\})\}$$

is greater than S in the partial ordering on $\mathfrak{S}$. This is a contradiction. So $E_n \sim 1$.

Replacing E_1 by $1 - \sum \{E_n | n \geq 2\}$ if necessary, we obtain a matrix base $\{E_n\}$ that satisfies the required conditions. On the one hand, we have

$$\|E_mAE_m\| = \text{lub} \{\|E_{ni}AE_{ni}\| | (n, i) \in N_m \times I\} \leq \alpha_m$$

for $m \geq 2$, and

$$\|E_mBE_n\| = \text{lub} \{\|E_{pi}BE_{qj}\| | (p, q, i, j) \in N_m \times N_n \times I \times I\} = 0$$

for $m, n \geq 2$ and B in $\mathfrak{X}$. On the other hand, we have that

$$\begin{aligned}
\|E_1 A E_m\| &\leq \left\| A \sum \{E_{ni} | (n, i) \in N_m \times I\} \right\| \\
&\leq \left\| \sum \left\{ A \sum \{E_{ni} | i \in I\} | n \in N_m \right\} \right\| \leq \left\| \sum \{A F_n | n \in N_m\} \right\| \\
&\leq \left\| \sum \{F_n A F_n | n \in N_m\} \right\| \leq \text{lub} \{\|F_n A F_n\| | n \in N_m\} \leq \alpha_m
\end{aligned}$$

for $m \geq 2$. Q.E.D.

Now we can prove our main theorem.

THEOREM 3.8. *An element A in a properly infinite von Neumann algebra $\mathcal{Q}$ is a selfadjoint commutator if and only if 0 is in the essential central range of A modulo the strong radical of $\mathcal{Q}$.*

PROOF. If A is a selfadjoint commutator, then the essential central spectrum $\mathfrak{K}(A)$ of $\mathcal{Q}$ modulo the strong radical $\mathfrak{J}$ contains 0 . The proof presented in Theorem 3.4 is sufficient.

Conversely, we may assume $\mathcal{Q}$ has no σ -finite type I direct summands (Theorem 3.4). Let A be in $\mathcal{Q}$ and let $0 \in \mathfrak{K}(A)$. We have already indicated after Lemma 3.5 that with regard to a matrix base E_i ($1 \leq i \leq 4$) the operator A can be written as a 4×4 -matrix (A_{ij}) over $\mathcal{Q}$ such that $A_{41} = A_{32} = 0$, and such that A_{33} and A_{44} can be written with respect to an infinite matrix base as matrices over $\mathcal{Q}$ of the form (C_{ij}) , where $\|C_{ij}\| \leq \alpha_j \delta_{ij}$ for $i, j = 1, 2, \dots$. Let A' (resp. A'') be given by

$$A' = \begin{pmatrix} A_{11} & A_{14} \\ 0 & A_{44} \end{pmatrix} \left(\text{resp. } A'' = \begin{pmatrix} A_{22} & A_{23} \\ 0 & A_{33} \end{pmatrix} \right).$$

Setting C equal to A' (resp. A'') and applying Lemma 3.7 to C^* , we get a matrix base $\{F_n | 1 \leq n < \infty\}$ for $\mathcal{Q}$ such that $F_n B F_m = 0$ for all $2 \leq m < n < \infty$ and B in the set $\mathfrak{X} = \{1, C, C^*, C^* C, C C^*\}$, and $\|F_m C F_1\| \leq \alpha_m$, $\|F_m C F_m\| \leq \alpha_m$ for $m \geq 2$. From Lemma 3.6 applied to the matrix base $\{F_n\}$, we see that C may be written as a 4×4 -matrix over $\mathcal{Q}$ of the form Figure 1, where $\|A_i\|$, $\|B_i\|$, $\|C_i\|$, $\|D_i\|$ are majorized by α_i . Actually, the entries (*) of the 3,1 and 4,2 (resp. 3,4 and 4,3) blocks have the same form as the 4,1 (resp. 4,4) block. However, we do not need this information here. Reassembling the matrix A from the various parts and applying J. Anderson's rearrangement [1], viz. $1 \rightarrow 7 \rightarrow 6 \rightarrow 4 \rightarrow 1$, $2 \rightarrow 5 \rightarrow 2$, $3 \rightarrow 3$, $8 \rightarrow 8$, we obtain an 8×8 -matrix over $\mathcal{Q}$ whose four 2×2 -diagonal blocks have the form Figure 2, where $\|B_i\| \leq \alpha_i$, $\|C_i\| \leq \alpha_i$. From the calculation of M. David [5], we get that each of these four diagonal blocks is a selfadjoint commutator in $\mathcal{Q}$ and thus that A itself is a selfadjoint commutator (cf. [1], [11, Proof, 4]). Q.E.D.

*	*	*	0
*	*	0	*
*	B_1 B_2 $\vdots$	C_1 C_2 $\ddots$ 0	*
A_1 A_2 $\vdots$	*	*	D_1 D_2 $\ddots$ 0

$$\left(\begin{array}{c|ccc} * & & & 0 \\ \hline B_1 & C_1 & & 0 \\ B_2 & & C_2 & \\ \vdots & & & \ddots \\ & 0 & & \end{array} \right)$$

FIGURE 2

An operator A in a von Neumann algebra $\mathcal{Q}$ is said to be *similar* to an operator B in $\mathcal{Q}$ if there is an invertible C in $\mathcal{Q}$ with $CAC^{-1} = B$. We can now characterize those operators in a properly infinite algebra that are similar to a selfadjoint commutator.

PROPOSITION 3.9. *An operator A in a properly infinite von Neumann algebra $\mathcal{Q}$ with strong radical $\mathcal{J}$ is similar to a selfadjoint commutator in $\mathcal{Q}$ if and only if there is a $\kappa > 0$ such that*

$$\|(\alpha - A)(\mathcal{J} + [\zeta])\| \geq \kappa|\alpha|$$

for every complex α and every ζ in the spectrum Z of the center $\mathcal{Z}$ of $\mathcal{Q}$, or equivalently, such that

$$\|(\alpha - A)(\mathfrak{M})\| \geq \kappa|\alpha|$$

for every α and maximal ideal $\mathfrak{M}$ in $\mathcal{Q}$.

PROOF. Suppose there exists such a $\kappa > 0$. For convenience suppose that $\|A\| = 1$. There is a C in $\mathcal{Z}$ such that

$$C^\wedge(\zeta) = \|A(\mathcal{J} + [\zeta])\|$$

[12, 3.2]. There is a projection P in $\mathcal{Z}$ such that $CP > 0$ (i.e. $\{\zeta \in Z | C^\wedge(\zeta) > 0\}$ is dense in $\{\zeta \in Z | P^\wedge(\zeta) = 1\}$) and $C(1 - P) = 0$. We have that $A(1 - P)$ is in the strong radical of $\mathcal{Q}_{(1-P)}$ [12, 3.1], and thus $A(1 - P)$ is a selfadjoint commutator in $\mathcal{Q}_{(1-P)}$. By passing to the direct summand $\mathcal{Q}_P$, we may assume that $C > 0$. There is an orthogonal sequence $\{P_n\}$ of projections in $\mathcal{Z}$ of sum 1 such that $C^\wedge(\zeta)$ is in the interval $[(n+1)^{-1}, n^{-1}]$ whenever $P_n^\wedge(\zeta) = 1$. For every $n = 1, 2, \dots$ there is a B_n in $\mathcal{J}$ such that

$$\|(A + B_n)([\zeta])\|$$

is in the interval $[(n+1)^{-1}, 2n^{-1}]$ for every ζ with $P_n^\wedge(\zeta) = 1$. The sum $\sum (n+1)(A + B_n)P_n$ converges in the strong topology to an element B in $\mathcal{Q}$. We have that

$$\|B(\mathcal{J} + [\zeta])\| \geq 1$$

and

$$\|(\alpha - B)(\mathcal{J} + [\zeta])\| \geq \kappa|\alpha|$$

for every complex α and every ζ in the union of the sets $\{\zeta \in Z | P_n^\wedge(\zeta) = 1\}$ and thus for every ζ in Z . This means that the canonical image of B in the algebra $\mathcal{Q}/(\mathcal{J} + [\zeta])$ is never a scalar multiple of the identity. There is an invertible C in $\mathcal{Q}$ and a projection F equivalent to 1 such that $FCBC^{-1}F = 0$ [11, Proof, Theorem 4]. For every $\varepsilon > 0$ and every $n = 1, 2, \dots$, there is a projection E_n in $\mathcal{J}$ such that

$$\|B_n(1 - E_n)\| \leq \varepsilon/(\|C\| \|C^{-1}\|).$$

The range projection F_n of $F(C^{-1})^*E_n$ is in $\mathcal{J}$ and thus $G_n = F - F_n$ is equivalent to 1. Setting $G = \sum G_n P_n$, we get a projection G equivalent to 1 such that

$$\|GCAC^{-1}G\| \leq \|FCBC^{-1}F\| + \|C\| \|C^{-1}\| \text{lub}_n \|B(1 - E_n)\| \leq \varepsilon.$$

We conclude that $\mathcal{K}(CAC^{-1})$ contains 0 (Theorem 2.2), and consequently that CAC^{-1} is a selfadjoint commutator in $\mathcal{Q}$ (Theorems 3.4 and 3.8).

Conversely, let A be a selfadjoint commutator in $\mathcal{Q}$. We have that

$$(6) \quad \|(\alpha - A)(\mathcal{J} + [\zeta])\| \geq |\alpha|$$

for every scalar α and every $\zeta \in Z$; otherwise,

$$|\alpha| = |\phi(\alpha - A)(\zeta)| \leq \|(\alpha - A)(\mathcal{J} + [\zeta])\| < |\alpha|$$

for some α and ζ . Here ϕ is a state of $A^\sim$ which vanishes on $\mathcal{J} \cup \{A\}$ (Theorems 3.4 and 3.8). If C is an invertible element in $\mathcal{Q}$, then we have that

$$\|(\alpha - CAC^{-1})(\mathcal{J} + [\zeta])\| \geq (\|C\| \|C^{-1}\|)^{-1} |\alpha|$$

for every α and ζ . Q.E.D.

REMARK. The preceding proposition shows that the operators of class (F') in a properly infinite algebra, viz. those operators which are not equal to a scalar multiple of the identity (zero included) modulo any maximal ideal, that were studied in [10] are contained in the set of operators similar to a selfadjoint commutator.

COROLLARY 3.10. *For any operator A and any normal operator B in a properly infinite von Neumann algebra the operator $BA - AB$ is a selfadjoint commutator.*

PROOF. We may assume that 0 is in the essential central spectrum of B modulo the strong radical [12, 3.5]. For every $\varepsilon > 0$, there is a projection E in $\mathcal{Q}$ equivalent to the identity with

$$\|EB\| = \|BE\| \leq \varepsilon$$

(Corollary 2.6). Hence, the essential central spectrum of $BA - AB$ modulo the strong radical contains 0 (Theorem 2.2) and so $BA - AB$ is a selfadjoint commutator (Theorem 3.8). Q.E.D.

REMARK. Actually, for any A and any seminormal B (i.e. either $\pm(BB^* - B^*B)$ is positive) in a properly infinite algebra, the operator $AB - BA$ is a selfadjoint commutator (cf. [11, Note "added in proof"]).

The next proposition extends the results of H. Radjavi [13] for the algebra of bounded operators on a separable Hilbert space.

PROPOSITION 3.11. *A selfadjoint operator in a properly infinite von Neumann algebra is a selfadjoint commutator if and only if its essential central spectrum modulo the strong radical contains a nonpositive and a nonnegative element.*

PROOF. The essential central range of a selfadjoint operator modulo the strong radical is the smallest central convex set containing the essential central spectrum modulo the strong radical [12, 4.4]. Q.E.D.

The next result extends that of J. Williams [14] for the algebra of all bounded operators on a separable Hilbert space.

PROPOSITION 3.12. *Every commutator of a properly infinite factor von Neumann algebra is similar to a selfadjoint commutator.*

PROOF. An operator A is a commutator in a properly infinite factor $\mathcal{A}$ if and only if there is an invertible B in $\mathcal{A}$ and a projection E in $\mathcal{A}$ with $E \sim 1$ such that $EBAB^{-1}E = 0$ [3], [4], [10]. Thus, every commutator of $\mathcal{A}$ is similar to a selfadjoint commutator (Theorems 3.4 and 3.8).

One may also use the arguments of J. Williams [14] together with Theorems 3.4 and 3.8 to avoid the computations of [3], [4], [10]. Q.E.D.

One might conjecture that an operator is a commutator if and only if it is not equal to a nonzero scalar multiple of the identity modulo every maximal ideal (i.e. if and only if it is an operator of class (F)). If the center is large, this turns out to be false.

EXAMPLE 3.13. In every properly infinite von Neumann algebra $\mathcal{A}$ not equal to a finite product of factors, there is an operator of class (F) which is not a commutator. Indeed, there is for every natural number m a neighborhood $V = V_m$ of 1 contained in the sphere of radius one about 1 such that $AB - BA$ in V_m and $\|A\| \leq 1$ implies $\|B\| \geq m$. This holds even for factors. Notice that every neighborhood of the identity contains commutators [10, 3.10]. On the contrary, there would be bounded sequences $\{A_n\}$ and $\{B_n\}$ in $\mathcal{A}$ with $\{A_n B_n - B_n A_n\}$ converging to the identity. This is impossible [16, Problem 183]. So such a neighborhood V_m exists.

Now let $\{P_m\}$ be an orthogonal sequence of nonzero central projections of $\mathcal{A}$ of sum 1. There is a commutator C_m in $\mathcal{A}P_m$ such that $\|C_m - 1\| \leq 1$ and $C_m = AB - BA$ for $\|A\| \leq 1$ implies $\|B\| \geq m^2$. Then the operator $C = \sum m^{-1} C_m$ is in class (F). In fact, let ζ be in the spectrum of the center. Either $\hat{P}_m(\zeta) = 1$ for some m or $\hat{P}_m(\zeta) = 0$ for all m . In the first case

$$C(\mathcal{J} + [\zeta]) = (BA - AB)(\mathcal{J} + [\zeta])$$

for some A and B in $\mathcal{A}P_m$. Here $\mathcal{J}$ is the strong radical. Hence C is not a nonzero scalar multiple of the identity modulo $\mathcal{J} + [\zeta]$. In the second case we have that

$$\|C(\xi + [\zeta])\| \leq \left\| C \left(1 - \sum \{P_n | n \geq m\} \right) ([\zeta]) \right\| \leq 2(m+1)^{-1}$$

and so $C(\xi + [\zeta]) = 0$. Now suppose $C = AB - BA$ for A, B in $\mathcal{Q}$. We may assume that $\|A\| \leq 1$. This means that $\|mBP_m\| \geq m^2$. This is impossible. So C is not a commutator.

In contradistinction to factors we show that not every commutator is similar to a selfadjoint commutator when the center is sufficiently large.

PROPOSITION 3.14. *Let A be an operator of class (F) in a properly infinite von Neumann algebra $\mathcal{Q}$; then there is a sequence $\{\beta_n\}$ of strictly positive numbers and a sequence $\{P_n\}$ of orthogonal central projections of sum 1 such that $\sum \beta_n AP_n$ converges in the strong topology to a commutator in $\mathcal{Q}$.*

PROOF. Let $\mathcal{J}$ be the strong radical of $\mathcal{Q}$ and let $P_0 = c_{\mathcal{J}}(A)$. There is a sequence $\{P_n | n \geq 1\}$ of orthogonal central projections of sum $1 - P_0$ such that $\|A(\xi + [\zeta])\| \geq n^{-1}$ for every ζ in the spectrum of the center of $\mathcal{Q}P_n$ [12, 3.1]. Therefore, the operators AP_n ($n \geq 1$) are of class (F') in $\mathcal{Q}P_n$. There is B_n and C_n in $\mathcal{Q}P_n$ with

$$AP_n = B_n C_n - C_n B_n$$

for all $n \geq 0$ [10]. Setting β_n equal to the inverse of $\|B_n\| \|C_n\| + 1$, we see that $\sum \beta_n AP_n$ converges in the strong topology and that it is a commutator in $\mathcal{Q}$. Q.E.D.

EXAMPLE 3.15. Let $\mathcal{Q}$ be a properly infinite von Neumann algebra. Suppose there is a sequence $\{P_n\}$ of nonzero orthogonal central projections of sum 1. Let E_n be a projection in $\mathcal{Q}P_n$ such that $E_n \sim P_n - E_n \sim P_n$ and let

$$A = \sum (n^{-1} P_n + n^{-2} E_n).$$

Then A is in class (F) and so there is a sequence $\{\beta_n\}$ of strictly positive numbers and a sequence $\{Q_n\}$ of orthogonal central projections of sum 1 such that $B = \sum \beta_n A Q_n$ is a commutator in $\mathcal{Q}$ (Proposition 3.14). There is no state ϕ in $\mathcal{Q}^\sim$ and invertible C in $\mathcal{Q}$ with $\phi(\mathcal{J}) = (0)$ and $\phi(CBC^{-1}) = 0$; otherwise, the state ϕ would vanish on CAC^{-1} or equivalently $\mathcal{K}(CAC^{-1})$ would contain 0. This is known to be impossible (cf. [11, p. 63ff.]). Hence, the operator B is not similar to a selfadjoint commutator.

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